AXISYMMETRIC PROBLEM OF HEAT AND MASS TRANSFER IN SATURATED POROUS MEDIUM

N. N. Verigin, I. L. Khabibullin, and G. A. Khalikov

UDC 536.01:532.546:517.947

Self-similar solutions are obtained for the system of heat-conduction and filtration equations in regions separated by a moving boundary in the axisymmetric case.

Mathematical models of problems regarding thermal methods of intensifying the extraction of useful minerals — high-viscosity oils, bitumen, sulfur, and gas hydrates — are, under certain assumptions, complex variants of the Stefan problem [1-4]. Analogous heat- and mass-transfer problems described by heat-conduction and filtration equations are also encountered in theories of drying [5] and the destruction of solid materials [6-8]. In [1-4, 6-8], consideration was given to one-dimensional linear problems in regions separated by a moving phase-transition surface (melting of solid phase, evaporation of liquid, breakdown of gas hydrates, etc.). In the present work, a similar problem is considered in the case of plane—radial geometry. In contrast to the above-mentioned works, the liquid-phase filtration is nonsteady, and the pressure at the phase-transition surface is not specified, but is determined from additional considerations.

Suppose that in some small region (e.g., in a borehole) of radius r_c of a porous medium there exists a heat source. Then the temperature of the surrounding medium increases with time, and the solid material filling the pores changes its aggregate state, e.g., melts or sublimes. Filtration of the new liquid or gaseous phase occurs in the direction toward the heat source, and the melting surface R(t) descends into the porous medium. Thus, two regions are formed in the medium: the region of filtration of the new phase, $r_c < r < R(t)$; and the initial state, $R(t) < r < \infty$.

The temperature distribution in the medium in the case of constant thermophysical parameters is described by the following system of equations

$$\frac{\lambda_I}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_I}{\partial r}\right)-\rho_s c_1 v\frac{\partial T_I}{\partial r}=\alpha_I\frac{\partial T_I}{\partial t}, \tag{1}$$

$$\frac{\lambda_{II}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_{II}}{\partial r}\right)=\alpha_{II}\frac{\partial T_{II}}{\partial t}, \tag{2}$$

where $\lambda_I=\lambda_0(1-m)+\lambda_1 m;\ \lambda_{II}=\lambda_0(1-m)+\lambda_2 m;\ \alpha_I=\rho_0 c_0(1-m)+\rho_1 c_1 m;\ \alpha_{II}=\rho_0 c_0(1-m)+\rho_2 c_2 m$ are the mean thermal conductivities and specific heats of regions I — $r_c < r < R(t)$ — and II — $R(t) < r < \infty$; ρ_s is the mean liquid-phase density; m is the saturation of the solid phase. Here and below, the subscripts 0, 1, and 2 refer to the pore body, the liquid phase, and the solid phase, respectively.

The filtration rate of the new phase is

$$v=-\frac{k}{\mu}\frac{\partial P}{\partial x}. \tag{3}$$

The pressure distribution in region I is described by the nonsteady-filtration equation

$$\frac{\varkappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)=\frac{\partial P}{\partial t}, \tag{4}$$

All-Union Scientific-Research Institute of Water Supply, Canalization, Hydrotechnical Construction and Engineering Hydrogeology, Fortieth-Anniversary-of-the-October-Revolution Pashkir State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 38, No. 5, pp. 913-918, May, 1980. Original article submitted June 5, 1979.

0022-0841/80/3805-0581$07.50

which is accurate for the liquid phase and approximate in the sense of Leibenzon linearization over the pressure P for the gas phase.

The following obvious boundary conditions may be adopted for Eqs. (1) and (2)

$$T_{\rm I}(0,\ t)=T_{\rm c},\ T_{\rm II}(r,\ 0)=T_{\rm II}(\infty,\ t)=T_0. \tag{5}$$

On surface R(t), temperature continuity and conditions of heat and mass balance are observed:

$$T_{\rm I}(R,\ t)=T_{\rm II}(R,\ t)\equiv T_{\rm m}, \tag{6}$$

$$-\lambda_{\rm I}\frac{\partial T_{\rm I}}{\partial r}+\lambda_{\rm II}\frac{\partial T_{\rm II}}{\partial r}=(c_1T_{\rm m}+L_2)\,m\rho_2\frac{dR}{dt}, \tag{7}$$

$$v(R,\ t)=-m\,\frac{\rho_2-\rho_1}{\rho_1}\,\frac{dR}{dt}. \tag{8}$$

The phase-transition temperature T_m is known. Equation (7) takes account of the heat-conduction heat flux in the two regions, the convective heat transfer in the first region and the latent heat of the phase change L_2. This condition is preliminarily transformed taking Eq. (8) into account. The balance relations in Eqs. (7) and (8) are more rigorous than the analogous conditions given in the above-mentioned works, since in Eqs. (7) and (8) account is taken of the relative motion of the new phase and the surface R(t).

On the phase-transition surface, the following condition is assumed for the pressure

$$P(R,\ t)=P_{\rm m}=f(T_{\rm m}). \tag{9}$$

The condition is none other than the phase-equilibrium relation, which always holds in phase transitions of the first kind. The form of this function is determined experimentally [4] or else theoretically, e.g., from the Clapeyron–Clausius equation. If there is no sharp phase transition on the surface R(t), a change in aggregate state of amorphous media (bitumens, asphaltenes, etc.) is observed, and the initial pressure in the medium may be taken as P_m. This case is possible, as a rule, in problems of useful-mineral exploitation and the so-called rock pressure may be taken as the initial pressure.

The problem in Eqs. (1)-(9) is closed and admits of self-similar solution. The difference between this problem and those known in the literature is that the hydrodynamic problem is "semiinverse in the boundary conditions," i.e., the pressure and production of the new phase at $r = r_c$ is not specified but is determined from the solution.

The solution of Eq. (4) is sought in the form [5]

$$P=A_1\,{\rm Ei}\left(-\frac{r^2}{4\varkappa t}\right)+B_1, \tag{10}$$

where $\mathrm{Ei}(-\xi)=-\int\limits_{\xi}^{\infty}\exp(-u)\,\frac{du}{u}$ is an integral–exponential function.

Determining the constants A_1 and B_1 from Eqs. (8) and (9) gives

$$P(r,\ t)=P_{\rm m}+\frac{m\mu}{k}\,\frac{\rho_2-\rho_1}{\rho_1}\,\beta\exp\left(\frac{\beta}{4\varkappa}\right)\left[{\rm Ei}\left(-\frac{r^2}{4\varkappa t}\right)-{\rm Ei}\left(-\frac{\beta}{4\varkappa}\right)\right], \tag{11}$$

where

$$\beta=R^2(t)/t={\rm const}. \tag{12}$$

From Eq. (11), the flow rate of new phase at the outlet from the porous medium may be determined

$$Q=2\pi hr_{\rm c}\,\frac{k}{\mu}\,\frac{\partial P(r_{\rm c},\ t)}{\partial r}=4\pi hm\,\frac{\rho_2-\rho_1}{\rho_1}\,\beta\exp\left(\frac{\beta}{4\varkappa}-\frac{r_{\rm c}^2}{4\varkappa t}\right) \tag{13}$$

and also the pressure (at $r = r_c$).

Substituting the value of the filtration rate according to Eqs. (11) and (3) into Eq. (1), and introducing the self-similar variables $z_1 = r^2/4a_{\mathrm{I}} t$, $z_2 = r^2/4a_{\mathrm{II}} t$, Eqs. (1) and (2) are rewritten as ordinary differential equations

$$\frac{d^2T_{\mathrm{I}}}{dz_1^2} + \left[1 + \frac{1}{z_1}\left(1 + \theta \exp\left(-\frac{z_1}{\nu}\right)\right)\right]\frac{dT_{\mathrm{I}}}{dz_1} = 0,\ \nu = \frac{\varkappa}{a_{\mathrm{I}}},$$
$$a_i = \frac{\lambda_i}{\alpha_i},$$
$$\frac{d^2T_{\mathrm{II}}}{dz_2^2} + \left(1 + \frac{1}{z_2}\right)\frac{dT_{\mathrm{II}}}{dz_2} = 0,\ \theta = \frac{(\rho_2 - \rho_1)\, c_1 m}{\alpha_{\mathrm{I}}\, a_{\mathrm{I}}}\,\beta \exp\frac{\beta}{4\varkappa}. \quad (14)$$

These equations are solved, taking Eqs. (5) and (6) into account, to give

$$T_{\mathrm{I}} = T_{\mathrm{m}} + (T_c - T_{\mathrm{m}})\frac{N(z_1)}{N(0)},$$
$$N(\xi) = \int_{\xi}^{\gamma_1} \exp\left[\theta \,\mathrm{Ei}\left(-\frac{u}{\nu}\right) - u\right]\frac{du}{u}, \quad (15)$$

$$T_{\mathrm{II}} = T_0 + (T_{\mathrm{m}} - T_0)\frac{\mathrm{Ei}(-z_2)}{\mathrm{Ei}(-\gamma_2)},\quad \gamma_1 = \frac{R^2(t)}{4a_{\mathrm{I}}t} = \mathrm{const}, \quad (16)$$
$$\gamma_2 = \frac{R^2(t)}{4a_{\mathrm{II}}t} = \mathrm{const}.$$

The law of motion of the mobile boundary may be determined from the second relation in Eq. (16)

$$R(t) = 2\sqrt{a_{\mathrm{I}}\,\gamma_1 t}. \quad (17)$$

The problem is completely solved if the constants β, γ_1, and γ_2 are determined. It follows from Eqs. (12) and (16) that

$$\beta = 4\gamma_1 a_{\mathrm{I}},\ \gamma_2 = \gamma_1\frac{a_{\mathrm{I}}}{a_{\mathrm{II}}}.$$

Substituting Eqs. (15)-(17) into Eq. (7), and taking the last two formulas into account, a transcendental equation for determining γ_1 is obtained:

$$\lambda_{\mathrm{I}}(T_c - T_{\mathrm{m}})\frac{\exp\left[\theta\,\mathrm{Ei}\left(-\frac{\gamma_1}{\nu}\right) - \gamma_1\right]}{N(0)} + \lambda_{\mathrm{II}}(T_{\mathrm{m}} - T_0)\frac{\exp\left(-\gamma_1\frac{a_{\mathrm{I}}}{a_{\mathrm{II}}}\right)}{\mathrm{Ei}\left(-\gamma_1\frac{a_{\mathrm{I}}}{a_{\mathrm{II}}}\right)} = (c_1 T_{\mathrm{m}} + L_2)\, m\rho_2 a_{\mathrm{I}}\gamma_1, \quad (18)$$
$$\theta = \frac{4mc_1(\rho_2 - \rho_1)}{\alpha_{\mathrm{I}}}\gamma_1 \exp\left(\frac{\gamma_1}{\nu}\right).$$

For porous media, as a rule, $\varkappa \gg a_{\mathrm{I}}$, i.e., the pressure perturbation travels more rapidly than the temperature perturbation [4]. Therefore, the liquid-phase filtration may be regarded as quasisteady, and the above equations then simplify.

TABLE 1. Change with Time in the Melting-Surface Coordinate and the Pressure Difference in the First Region

$\frac{t}{\tau}$	1	10	50	100	500	1000
			$R(t)\cdot 10^2$, m			
I	5.1	16.1	36.1	50.9	113.8	161.2
II	5.9	18.6	41.6	58.8	131.5	186.2
III	7.1	22.3	49.9	70.6	157.7	223
			$-\Delta P$, atm			
I	0.08	0.16	0.24	0.27	0.36	0.39
II	0.12	0.23	0.34	0.38	0.50	0.55
III	0.19	0.37	0.52	0.59	0.75	0.83

Solving the boundary problem in Eqs. (4), (8), and (9) in the quasisteady case (setting $\partial P/\partial t = 0$) leads to the result

$$P = P_m - \frac{m\mu}{k}\,\frac{\rho_2-\rho_1}{\rho_1}\,\beta\ln\frac{R^2}{r^2}. \tag{19}$$

The filtration rate is given by the expression

$$v = -m\,\frac{\rho_2-\rho_1}{\rho_1}\,\beta\,\frac{2}{r},$$

and then the exponential term drops out of Eq. (14). Therefore, for the same boundary condition, the temperature distribution is described by the formula

$$T_I = T_m + (T_c - T_m)\left[1-\frac{\Gamma(\theta, z_1)}{\Gamma(\theta, \gamma_1)}\right],$$

where

$$\Gamma(\theta, \xi) = \int_0^{\xi}\exp(-u)\,u^{\theta-1}\,du.$$

In the case of quasisteady filtration, Eqs. (16) and (17) remain in force, and a transcendental equation simpler than Eq. (18) is obtained for determining γ_1

$$\lambda_I(T_c-T_m)\,\frac{\exp(-\gamma_1)\,\gamma_1^{\theta}}{\Gamma(\theta, \gamma_1)} + \lambda_{II}(T_m-T_0)\,\frac{\exp\left(-\gamma_1\dfrac{a_I}{a_{II}}\right)}{\mathrm{Ei}\left(-\gamma_1\dfrac{a_I}{a_{II}}\right)} = (c_1T_m+L_2)\,m\rho_2a_I\gamma_1,\quad \theta = \frac{4mc_1(\rho_2-\rho_1)}{a_I}\,\gamma_1. \tag{20}$$

Consider, as an example, the melting of a bitumen layer in a porous medium. Values characteristic for bituminous deposits were assumed in the calculations: $\rho_0 = 2500$ kg/m^3; $\rho_1 = 800$ kg/m^3; $\rho_2 = 950$ kg/m^3; $c_0 = 750$ J/kg·deg; $c_1 = c_2 = 2000$ J/kg·deg; $\lambda_0 = 1.38$ W/m·deg; $\lambda_1 = 0.17$ W/m·deg; $\lambda_2 = 0.5$ W/m·deg; $L_2 = 1.67\cdot 10^3$ J/kg; $a_I = 0.625\cdot 10^{-6}$ m^2/sec; $a_{II} = 0.708\cdot 10^{-6}$ m^2/sec; $T_m = 60$°C; $k = 5\cdot 10^{-14}$ m^2; $\mu = 0.25$ N·sec/m^2; $\varkappa = 0.67\cdot 10^{-4}$ m^2/sec.

Values of 0.012, 0.016, and 0.022 were calculated for γ_1 from Eq. (18) for temperature differences $T_c - T_m$ in the first region of 40, 90, and 180°C, respectively.

Knowing γ_1, all the basic parameters of the investigated process can be deduced from the expressions given above: the temperature and pressure fields in the medium; the liquid-phase filtration rate; the melting-surface coordinate; the pressure at the coordinate origin; and the liquid-phase flow rate.

Table 1 gives values of the melting-surface coordinate R(t) and the pressure difference in the liquid-phase region — $\Delta P = P(r_c, t) - P_m$ — as a function of the time, as calculated from Eqs. (17) and (11), respectively (at $r = r_c$). In Table 1, $\tau = 86{,}400$ sec, and rows I, II, and III correspond, respectively, to the values 0.012, 0.016, and 0.022 of γ_1 given above. It follows from Table 1 that, with time, the motion of the surface R(t) rapidly slows down, and the pressure difference in the first region increases, this increase being more pronounced at the beginning of the heating process.

NOTATION

r, coordinate; t, time; T, temperature; P, pressure; ρ, density; λ, thermal conductivity; c, specific heat; a, thermal diffusivity; k, permeability; m, porosity; μ, viscosity; v, filtration rate; $\varkappa$, piezoconductivity; R(t), moving melting surface; L_2, latent heat of fusion of solid phase; T_m, melting point; z, z_1, z_2, self-similar variables; β, γ_1, γ_2, constants; A_1, A_2, constants of integration; ξ, u, auxiliary variables; α, mean specific heat; r_c, borehole radius; h, layer thickness.

LITERATURE CITED

1. N. N. Verigin, "Some problems of convective heat conduction in porous media," in: Proceedings of the All-Union Scientific-Research Institute of Water Supply, Canalization, Hydrotechnical Construction and Engineering Hydrogeology [in Russian], No. 9, Moscow (1964).
2. N. V. Cherskii and É. A. Bondarev, "Thermal method of exploiting gas-hydrate deposits," Dokl. Akad. Nauk SSSR, 203, No. 3 (1972).
3. N. N. Verigin and V. S. Golubev, "Vapor generation in underground collector beds," Dokl. Akad. Nauk SSSR, 223, No. 6 (1975).
4. E. B. Chekalyuk, Oil-Bed Thermodynamics [in Russian], Nedra, Moscow (1965).
5. A. V. Lykov, Heat and Mass Transfer [in Russian], Énergiya, Moscow (1972).
6. Yu. M. Shekhtman, "Heat-conduction problems in the structural breakdown of a material, taking into account the effect of escaping gas," Inzh. Zh., 5, No. 3 (1965).
7. K. K. Vasilevskii, "Self-similar problem of the heating of a semiinfinite body, taking account of mass loss and physicochemical changes within the material," Teplofiz. Vys. Temp., 5, No. 14 (1967).
8. I. G. Portnov, "Problems of destruction and melting, taking stepwise density change into account," in: Heat and Mass Transfer [in Russian], Vol. 2, Minsk (1968).

SOME PROBLEMS OF HEAT- AND MASS-TRANSFER THEORY SOLVABLE BY MEANS OF LAPLACE TRANSFORMATION

S. Ts. Koprinski

UDC 536.24.02

The solution of a system of heat- and mass-transfer equations is obtained in Laplace transforms; formulas for finding the inverse transforms are given.

Consider the system of heat- and mass-transfer equations [1]

$$\frac{\partial u}{\partial t} = a_1 \frac{\partial^2 u}{\partial x^2} + k_1 \frac{\partial^2 v}{\partial x^2},$$
$$\frac{\partial v}{\partial t} = a_2 \frac{\partial^2 v}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial x^2}, \qquad (1)$$

where $a_1 > 0;\ a_2 > 0;\ k_1 > 0;\ k_2 > 0;\ a_1 a_2 > k_1 k_2$.

It is required to find the solution of this system for which boundedness conditions are satisfied: $u(x,\ t) = 0(e^{\lambda_1 x}),\ \lambda_1 > 0;\ v(x,\ t) = 0(e^{\lambda_2 x}),\ \lambda_2 > 0;\ (0 \leqslant x < \infty)$.

Sofia, Bulgarian People's Republic. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 38, No. 5, pp. 919-921, May, 1980. Original article submitted September 20, 1977.